

The Implicit Function Theorem and the Global Methods of Cohomology

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1. INTRODUCTION

Let B be a real Banach space (possibly finite-dimensional) with the metric topology, let R^m be m -dimensional Euclidean space, let \mathcal{O} be a neighborhood of $(0, 0)$ in $R^m \times B$ that is homeomorphic to $R^m \times B$, and let $F: \mathcal{O} \rightarrow B$ be a continuous map. Assume $F(0, 0) = 0$. Let $\Omega = \{(\lambda, u) \in \mathcal{O} \mid F(\lambda, u) = 0\}$. Many analytical tools have been found useful in describing the local nature of Ω near the point $(0, 0)$, foremost of which is the implicit function theorem. This note concerns a global conclusion for the implicit function theorem. The conclusion in this form has been found useful for studying stability and bifurcation problems. Let $D_u F(\lambda, u)$ denote the differential of F with respect to u , that is, the Jacobian. The usual implicit function theorem assumes $D_u F(\lambda, u)$ exists and is nonsingular and continuous for all (λ, u) in some neighborhood of $(0, 0)$. The conclusion is local. The point of this paper is that a global conclusion is possible with no additional hypotheses.

Weakening slightly the main hypothesis of the implicit function theorem, we assume here that $D_u F(0, 0)$ exists and is nonsingular.

For B finite-dimensional, we consider all such (continuous) F . If B has infinite dimension, it is more usual to consider the fixed point set of a mapping. Thus, we assume $F(\lambda, u) = u - G(\lambda, u)$ where (a) $G(\lambda, u): \{\lambda\} \cap \mathcal{O} \rightarrow B$ is compact (i.e., takes bounded sets to sets with compact closure) for each λ , and (b) G is locally uniformly continuous in λ for bounded sets in B . Thus we are considering the fixed point set of a compact map which depends on a parameter. Conditions (a) and (b) are enough that $G: \mathcal{O} \rightarrow B$ is compact, which is what we actually

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use. In particular, since Ω is a subset of the image of G , it is locally compact.

For an arbitrary locally compact space X , let X^+ denote $X \cup \{\infty\}$, the one-point compactification of X . If X is compact, ∞ is an isolated point in X^+ . Let Ω_0 denote the connected component of Ω that contains $(0, 0)$, and let Ω_0^+ denote $(\Omega_0)^+$. Recall that two continuous functions $h_0, h_1: X \rightarrow Y$ are homotopic if there exists a continuous $H: X \times [0, 1] \rightarrow Y$ such that $H(\cdot, 0) = h_0$ and $H(\cdot, 1) = h_1$. A continuous function $h: X \rightarrow Y$ is said to be *inessential* if it is homotopic to a constant function; otherwise it is *essential*. Let S^m denote the m -dimensional sphere. Our result is the following.

THEOREM. *There is an essential map $f: \Omega_0^+ \rightarrow S^m$. Furthermore, $f|(\Omega_0^+ - \{(0, 0)\})$ is inessential.*

We will construct the map f in Section 2. The second statement will be immediate, since $f|(\Omega_0^+ - \{(0, 0)\})$ will not be surjective. The first statement is a consequence of the following stronger statement, which will be proved in Sections 3 and 4.

THEOREM. *The map f induces a nonzero map of m -dimensional Čech cohomology.*

This result for finite-dimensional B would not perhaps have been surprising to topologists studying dimension and degree arguments thirty or forty years ago, yet we have found no mention of the result. If Ω_0 is compact, the same result obtains if Ω_0^+ is replaced by Ω_0 . If Ω_0 is not compact, mention of the point at infinity can be eliminated by speaking of proper maps and proper homotopies. It is technically more convenient, however, to state the result as above. Čech cohomology must be used since Ω_0 need not be a very reasonable set. Away from $(0, 0)$, there are no conditions on the local structure of Ω_0 , aside from the fact that it is locally closed. For example, using a translate of the graph of $\sin(1/x)$, an Ω_0^+ can be constructed (for $m = \dim B = 1$) that is not arcwise connected and that carries no singular cohomology.

Rabinowitz [10] has proved a result for the case $m = 1$ which is related in spirit to our result. See also [1–3, 11–13] for related results and techniques.

To make the result seem reasonable, consider the case $m = 1$. Then near $(0, 0)$, the set Ω_0 looks like an interval and in particular the removal of $(0, 0)$ locally disconnects Ω_0^+ . Our theorem says, however, that the removal of $(0, 0)$ does not globally disconnect Ω_0^+ . If $m > 1$,

removal of $(0, 0)$ does not locally disconnect Ω_0^+ , but something can still be said. Our result is one way of saying it.

As an application, we consider the following. Let $B = R^n$. The stronger theorem and Alexander duality imply:

$$\text{rank } H_{n-1}(R^{m+n} - \Omega_0) \neq 0.$$

This in turn implies the following result ($n = 1$), which was proved in [14] under a stronger global hypothesis, and was used for the study of Liapunov functions.

COROLLARY. *Let $F: R^{m+1} \rightarrow R$ be C^1 . Assume for some $x_0 \in R^{m+1}$ that $F(x_0) = 0$ and that $\text{grad } F(x_0) \neq 0$. Then $R^{m+1} - F^{-1}(0)$ has at least two components and it has two components which intersect every neighborhood of x_0 .*

We remark that there are several more or less immediate generalizations of the result. For example, consider $F: R^m \times M \rightarrow R^n$, where M is an n -dimensional manifold. One supposes the hypotheses are satisfied at some point $(0, m_0) \in R^m \times M$. The correct analogous conclusion is: *There exists a neighborhood \mathcal{U} of $(0, m_0)$ and a map $F: (\Omega_0, \Omega_0 \cap (M - \mathcal{U})) \rightarrow (S^m, \{\infty\})$ which induces a nonzero map in m -dimensional Čech cohomology.* Another generalization which is valid is to consider $F: R^m \times R^n \rightarrow M$, where M is an arbitrary connected *noncompact* n -dimensional manifold. Also, the Jacobian condition can be relaxed. For example, in the finite dimensional case, the conclusion is true if the local degree of $F(0, u)$ at $u = 0$ is nonzero.

Finally, a word about the chronology of the result. It is essentially due to the second-named author and dates from the early 1970's. This author would like to thank C. C. Conley and R. A. Holzinger for helpful conversations during that period. Subsequently, the first-named author was able to streamline the arguments and to materially generalize the result. The present article is the outcome.

2. THE CONSTRUCTION OF THE MAP

The hypothesis on the Jacobian implies that there is a positive number $\eta < 1$ such that: $F(0, u) \neq 0$ for $0 < |u| \leq \eta$. Let S_η denote the set $\{(0, u) \mid |u| = \eta\}$. We topologize $(R^m \times B) \cup \{\infty\}$ so that complements of bounded sets form a neighborhood basis of ∞ . Note that if X is a locally compact closed subset of $R^m \times B$, the induced topology on $X \cup \{\infty\}$ is that of the one-point compacti-

fication. In particular, we can identify the subspace $R^m \cup \{\infty\}$ ($= (R^m \times \{0\}) \cup \{\infty\}$) with S^m and $\Omega \cup \{\infty\}$ with Ω^+ .

Suppose first that $\mathcal{O} = R^m \times B$. We consider the map

$$e: ((R^m \times B) \cup \{\infty\}) - S_n \rightarrow S^m$$

constructed as follows. For each point $x = (\lambda, u)$ in $(R^m \times B) - R^m$ (i.e., with $u \neq 0$), there is a unique closest point $\xi = \xi(x)$ on S_n , viz $(0, \eta | u |^{-1}u)$. The ray from ξ through x either impinges on R^m or not. If so, $e(x)$ is the point of impingement, and if not $e(x) = \infty$. On S^m , the map e is the identity. In formula,

$$\begin{aligned} e(\lambda, u) &= (\lambda \eta (\eta - |u|)^{-1}, 0) & \text{if } |u| < \eta, \\ &= \infty & \text{if } |u| \geq \eta, \\ e(\infty) &= \infty. \end{aligned}$$

The map $f: \Omega_0^+ \rightarrow S^m$ is the restriction $e | \Omega_0^+$. Clearly, $f^{-1}(0) = \{(0, 0)\}$; hence $f | (\Omega_0^+ - \{(0, 0)\})$ is not surjective. Since $\eta < 1$, we have $|e(x)| \geq |x|$.

It will be convenient to have a family of maps homotopic to e . For each r , $0 < r \leq \infty$, define $w_r: S^m \rightarrow S^m$ by

$$w_r(y) = \begin{cases} yr(r - |y|)^{-1} & \text{if } |y| < t < \infty, \\ \infty & \text{if } |y| \geq t, t < \infty, \\ y & \text{if } t = \infty, \end{cases} y \in R^m, \\ w_r(\infty) = \infty.$$

Define $e_r: S^{m+n} - S_n \rightarrow S^m$ by $e_r = w_r \cdot e$. Then all e_r are homotopic to $e = e_\infty$ and $e_r(x) = \infty$ if $|x| \geq r$.

Suppose now that $\mathcal{O} \neq R^m \times B$. By hypothesis, there is a homeomorphism $h: \mathcal{O} \rightarrow R^m \times B$. This homeomorphism h is used to translate everything to the case $\mathcal{O} = R^m \times B$. We will assume for the remainder of the proof that $\mathcal{O} = R^m \times B$ and leave to the reader refinements necessary if $\mathcal{O} \neq R^m \times B$.

It will sometimes be convenient to regard our maps as maps of pairs, e.g., $f: (\Omega_0^+, \{\infty\}) \rightarrow (S^m, \{\infty\})$. Such notation is standard and should not cause trouble.

3. THE PROOF IN THE FINITE-DIMENSIONAL CASE

If $B = R^n$, then $(R^m \times R^n) \cup \{\infty\} = S^{m+n}$, and S_n is an $(n-1)$ -dimensional sphere. Let $\tilde{H}_*(X)$ denote the reduced singular homology of X and $\tilde{H}^*(X)$ denote the Čech cohomology of X . Also let

$$i: S^m \rightarrow S^{m+n} - S_n, \quad \bar{i}: S_n \rightarrow S^{m+n} - \Omega^+$$

and $i_\Omega: \Omega \rightarrow S^{m+n} - S_n$ denote inclusions in $S^{m+n} - S_n$. We shall need the Alexander duality theorem [4, p. 301] which states: *Let X be a neighborhood retract in S^{m+n} (e.g., X is an open set or a submanifold). Then there is a natural duality isomorphism*

$$D: \check{H}^m(S^{m+n} - X) \rightarrow \check{H}_{n-1}(X)$$

for $m, n \geq 1$. In particular, letting $X = S_n$ on the left and $X = S^{m+n} - \Omega^+$ on the right, we have the following commutative diagram.

$$\begin{array}{ccccc} \check{H}^m(S^m) & \xrightleftharpoons[\iota_*]{e^*} & \check{H}^m(S^{m+n} - S_n) & \xrightarrow{i_\Omega^*} & \check{H}^m(\Omega^+) \\ & & \downarrow \approx D & & \downarrow \approx D \\ & & \check{H}_{n-1}(S_n) & \xrightarrow{i_*} & \check{H}_{n-1}(S^{m+n} - \Omega^+) \end{array}$$

The three left-hand groups are all isomorphic to the integers. Since $e: S^m \rightarrow S^m$ is the identity, $\iota^*e^* = 1$. Hence, e^* is an isomorphism. Let s denote a generator of $\check{H}^m(S^m)$ and σ denote a generator of $\check{H}_{n-1}(S_n)$. We have proved the following lemma.

LEMMA. *If $i_*(\sigma) \neq 0$ in $\check{H}_{n-1}(S^{m+n} - \Omega^+)$, then $i_\Omega^*e^*(s) \neq 0$ in $\check{H}^m(\Omega^+)$.*

Note that this lemma does not use the fact that $D_u F(0, 0)$ is nonsingular or that Ω is nonempty. It does require that Ω and S_n are disjoint.

We now show that $i_*(\sigma) \neq 0$. We quote the following result which is a form of the Borsuk-Ulam theorem: if $\mu: S^{n-1} \rightarrow R^n - \{0\}$ is such that $\mu(-x) = -\mu(x)$ for all $x \in S^{n-1}$, then

$$\mu_*: \check{H}_{n-1}(S^{n-1}) \rightarrow \check{H}_{n-1}(R^n - \{0\})$$

is nonzero ([9, p. 135], or for a more analytic proof [7, p. 91]). Since $D_u F(0, 0)$ is nonsingular, $D_u F(0, 0)|_{S_n}: S_n \rightarrow R^n - \{0\}$ is such a map. Let $\tilde{F}: R^{m+n} - \Omega \rightarrow R^n - \{0\}$ be $F|_{R^{m+n} - \Omega}$. Then $\tilde{F}|_{S_n}$ is homotopic in $R^n - \{0\}$ to $D_u F(0, 0)$ via the homotopy

$$\begin{aligned} F_h(p) &= D_u F(0, 0) \cdot p, & h &= 0, \\ &= h^{-1}F(0, hp), & h &\in (0, 1]. \end{aligned}$$

Thus, $\tilde{F}_*i_*(\sigma) \neq 0$; hence, $i_*(\sigma) \neq 0$.

Thus we have shown $f^* = (e|_{\Omega^+})^*: \check{H}^m(S^m) \rightarrow \check{H}^m(\Omega^+)$ is non-zero. Since f is homotopic to f_r for all r , $0 < r \leq \infty$, we have

$$f_r^*: \check{H}^m(S^m) \rightarrow \check{H}^m(\Omega^+)$$

is nonzero. The same is true in the relative case; that is, $\check{H}^m(S^m, \{\infty\})$ is isomorphic to the integers, with generator s' , say, and $f_r^*(s') \neq 0$ in $\check{H}^m(\Omega^+, \Omega^+ \cap \tau_r)$ where τ_r is defined below.

We have shown that $f_\Omega = e|_{\Omega^+}: \Omega^+ \rightarrow S^m$ induces a nonzero map in m -dimensional Čech cohomology. We must show the same is true for $f: \Omega_0^+ \rightarrow S^m$. To do this, we must use some deeper properties of Čech cohomology. The argument will not require that $\dim B$ is finite; hence, it will be valid in the infinite dimensional case as well.

Let us set some notation. Let

$$\begin{aligned}\beta_r &= \{x \in R^m \times B \mid |x| < r\}, \\ \bar{\beta}_r &= \{x \in R^m \times B \mid |x| \leq r\}, \\ \Sigma_r &= \{x \in R^m \times B \mid |x| = r\}, \\ \tau_r &= \{x \in R^m \times B \mid |x| \geq r\} \cup \{\infty\}.\end{aligned}$$

Since Ω_0^+ and $\Omega_0^+ \cap \tau_r$ are compact, the following is valid.

$$\check{H}^m(\Omega_0^+) = \check{H}^m(\Omega_0^+, \{\infty\}) = \text{direct lim } \check{H}^m(\Omega_0^+, \Omega_0^+ \cap \tau_r) \quad [5, \text{Section X. 3}].$$

We also have a commutative diagram:

$$\begin{array}{ccc} & & \check{H}^m(\Omega_0^+, \{\infty\}) \\ & \nearrow f^* & \parallel \\ \check{H}^m(S^m) = \check{H}^m(S^m, \{\infty\}) & \xrightarrow{f_r^*} & \check{H}^m(\Omega_0^+, \{\infty\}) \\ & \searrow f_r^* & \uparrow \\ & & \check{H}^m(\Omega_0^+, \Omega_0^+ \cap \tau_r)\end{array}$$

Hence, to show $f^*(s) \neq 0$ in $\check{H}^m(\Omega_0^+)$, we need to show

$$f_r^*: \check{H}^m(S^m, \{\infty\}) \rightarrow \check{H}^m(\Omega_0^+, \Omega_0^+ \cap \tau_r) \quad (*)$$

is nonzero for each r , $0 < r < \infty$.

To this end, let \bar{C}_0, \bar{C}_1 be a separation of $\Omega \cap \bar{\beta}_r$. That is, \bar{C}_0, \bar{C}_1 are open and closed in $\Omega \cap \bar{\beta}_r$, $\bar{C}_0 \cap \bar{C}_1 = \emptyset$, $\bar{C}_0 \cup \bar{C}_1 = \Omega \cap \bar{\beta}_r$. Suppose $(0, 0) \in \bar{C}_0$. Let $C_0 = \bar{C}_0 \cap \beta_r$, $C_1 = \bar{C}_1 \cap \beta_r$. Then

C_0^+ , C_1^+ are closed in $(\Omega \cap \beta_r)^+ = (\Omega \cap \beta_r)/(\Omega \cap \Sigma_r)$, and there is a Mayer-Vietoris sequence relating the cohomologies of the pairs

$$\begin{aligned} (C_0^+, \{\infty\}), (C_1^+, \{\infty\}), ((C_0 \cap C_1)^+, \{\infty\}) &= (\{\infty\}, \{\infty\}), \\ ((C_0 \cup C_1)^+, \{\infty\}) &= ((\Omega \cap \beta_r)^+, \{\infty\}). \end{aligned}$$

Since $\check{H}^*(\{\infty\}, \{\infty\}) = 0$, we have

$$\check{H}^m((\Omega \cap \beta_r)^+, \{\infty\}) = \check{H}^m(C_0^+, \{\infty\}) \oplus \check{H}^m(C_1^+, \{\infty\}).$$

Since $f_r|_{\bar{C}_1} : \bar{C}_1 \rightarrow S^m$ is not surjective, $(f_r|_{\bar{C}_1})^*(s) = 0$ in

$$\check{H}^m(\bar{C}_1, \bar{C}_1 \cap \Sigma_r) \approx \check{H}^m(C_1^+, \{\infty\}).$$

(Reference for this and similar isomorphisms of groups [5, Section X.5].) We know, however, that $(f_r|_{\Omega \cap \beta_r})^*(s') \neq 0$ in

$$\check{H}^m(\Omega \cap \bar{\beta}_r, \Omega \cap \Sigma_r) \approx \check{H}^m(\Omega^+, \Omega^+ \cap \tau_r) \approx \check{H}^m((\Omega \cap \beta_r)^+, \{\infty\}).$$

Therefore, $(f_r|_{\bar{C}_0})^*(s') \neq 0$ in

$$\check{H}^m(\Omega \cap \bar{\beta}_r, \Omega \cap \Sigma_r) \approx \check{H}^m((\Omega \cap \beta_r)^+, \{\infty\}).$$

If we consider all separations \bar{C}_0, \bar{C}_1 with $(0, 0) \in \bar{C}_0$, we find that $\cap \bar{C}_0 = (\Omega \cap \bar{\beta}_r)_0 =$ component of $(0, 0)$ in $\Omega \cap \bar{\beta}_r$ [8, Section II. 4, Proposition B]. In terms of pairs, $\cap (C_0^+, \{\infty\}) = (((\Omega \cap \beta_r)_0)^+, \{\infty\})$. Therefore,

$$\check{H}^m(((\Omega \cap \beta_r)_0)^+, \{\infty\}) = \text{direct lim } \check{H}^m(C_0^+, \{\infty\}) \quad [5, \text{Section X, 3}].$$

Hence, $(f_r|_{(\Omega \cap \bar{\beta}_r)_0})^*(s') \neq 0$ in

$$\check{H}^m(((\Omega \cap \beta_r)_0)^+, \{\infty\}) \approx H((\Omega \cap \bar{\beta}_r)_0, (\Omega \cap \beta_r)_0 \cap \Sigma_r).$$

We have the inclusion $j: (\Omega \cap \bar{\beta}_r)_0 \rightarrow \Omega_0 \cap \bar{\beta}_r$, and hence, the commutative diagram

$$\begin{array}{ccc} \check{H}^m(S^m, \{\infty\}) & \xrightarrow{f_r^*} & H^m(\Omega_0^+, \Omega_0^+ \cap \tau_r) \\ & \searrow & \Downarrow \\ & (f_r|_{\Omega_0 \cap \bar{\beta}_r})^* \circ j^* & \check{H}^m(\Omega_0 \cap \bar{\beta}_r, \Omega_0 \cap \Sigma_r) \xrightarrow{j_r^*} \check{H}^m((\Omega \cap \bar{\beta}_r)_0, (\Omega \cap \beta_r)_0 \cap \Sigma_r). \end{array}$$

The composition is nonzero; hence, clearly $f_r^*(s') \neq 0$. This proves (*), and the finite-dimensional case is done.

4. THE PROOF IN THE INFINITE-DIMENSIONAL CASE

Recall that $F(\lambda, u) = u - G(\lambda, u)$ for a compact G . The idea of this section is to approximate G by maps with finite-dimensional range to which we can apply the results of the last section. Then we use the continuity property of Čech cohomology again to nail down the result. Lest we be accused of running afoul of the counterexamples to the finite-rank approximation property [6], we point out that our approximating maps are definitely not linear.

We desire, for each sufficiently large integer N , a map

$$G_N: R^m \times B \rightarrow B$$

with the following properties:

- (i) $|G_N(\lambda, u) - G(\lambda, u)| \leq N^{-1}$ for $|(\lambda, u)| \leq N$,
- (ii) $G_N(0, u) \neq 0$ if $0 < |u| \leq \eta$,
- (iii) $\exists \epsilon = \epsilon(N) > 0$ such that $G_N(0, -u) = -G_N(0, u)$ if $|u| = \epsilon$,
- (iv) $\dim \text{range } G_N$ lies in a finite-dimensional subspace.

The details of constructing such G_N are rather standard. The idea is as follows. Note that $D_u G(0, 0): B \rightarrow B$ is compact. Suppose $\nu = \min\{|D_u G(0, 0)x| \mid x \in S_\eta\}$. For each $N > \nu^{-1}$, choose a finite set of points $P \subset S_\eta$ such that $-p \in P$ if $p \in P$, and such that every point in $D_u G(0, 0)(S_\eta)$ is within distance $\frac{1}{2} N^{-1}$ of some $D_u G(0, 0)(p)$. Using the Schauder idea, approximate $D_u G(0, 0)$ on S_η by H to within N^{-1} so that the image lies in the subspace generated by $D_u G(0, 0)(P)$, and so that $H(-x) = -H(x)$. Extend H to

$$\{x = (\lambda, u) \mid |x| \leq \eta\} \quad \text{by} \quad H(\lambda, \alpha u) = \alpha H(0, u)$$

if $0 \leq \alpha \leq 1$, $(0, u) \in S_\eta$. There exists $\epsilon = \epsilon(N) > 0$ such that $|H(x) - G(x)| < \frac{1}{2} N^{-1}$ if $|x| < 3\epsilon$. Use the Schauder idea again to approximate G on $\{x \in R^m \times B \mid 2\epsilon \leq |x| \leq N\}$ to within $\frac{1}{2} N^{-1}$ by a map H' with image contained in a finite-dimensional subspace. Use a continuous function $\varphi: [2\epsilon, 3\epsilon] \rightarrow [0, 1]$ with $\varphi(2\epsilon) = 0$, $\varphi(3\epsilon) = 1$ to amalgamate H and H' into H'' . Extend H'' to

$$G_N: R^m \times B \rightarrow B$$

by $G_N(\alpha x) = H''(x)$ if $\alpha \geq 1$, $|x| = N$. This G_N satisfies (i)–(iv).

Let

$$\begin{aligned}\Omega_N &= \{(\lambda, u) \mid G_N(\lambda, u) = u\}, \\ \Psi_N &= \{(\lambda, u) \mid \|G_N(\lambda, u) - u\| \leq N^{-1}\}, \\ \Phi_N &= \Omega^+ \cup \left(\bigcup_{M=N}^{\infty} \Psi_M \right).\end{aligned}$$

Let B_N be the range of G_N . Note that $\Omega_N \subset B_N$. If we consider $G_N \mid B_N$ we see by (ii), (iii) above that $G_N \mid B_N$ satisfies the conditions of the lemma of the previous section. Hence $(e \mid \Omega_N)^* s \neq 0$ in $\tilde{H}^m(\Omega_N)$. Since $\Omega_N \subset \Phi_N$ also $(e \mid \Phi_N)^* s \neq 0$ in $\tilde{H}^m(\Phi_N)$.

We now claim two facts: (1) each Φ_N is closed, and (2) every neighborhood of Ω^+ contains some Φ_N (note that $\Phi_N \supset \Phi_{N+1} \supset \cdots \supset \Omega^+$). Each Ψ_N is closed, so for the first, suppose there exists a sequence p_N, p_{N+1}, \dots converging to p with $p_M \in \Psi_M$. Then

$$\|G(p)\| = \lim \|G(p_M)\| \leq \lim (\|G(p_M) - G_M(p_M)\| + \|G_M(p_M)\|) = 0,$$

and $p \in \Omega$. The same type of argument establishes the second claim. By the continuity property of Čech cohomology [4, Appendix 3, Problem 3.16.3], $\tilde{H}^m(\Omega^+) = \text{direct lim } \tilde{H}^m(\Phi_N)$. (We must use this reference instead of [5, Chap. X], because [5, Chap. X] is for limits of compact sets.) Hence, $(e \mid \Omega^+)^* s = f_0^* s \neq 0$. We use the same trick as at the end of the last section to get from $f_0^* s \in \tilde{H}^m(\Omega^+)$ to

$$f^* s \in \tilde{H}^m(\Omega_0^+).$$

Thus, the proof is complete.

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